Evidence-Based Subjective Logic

Boris Škorić
Nicola Zannone
Sebastiaan de Hoogh

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http://arxiv.org/abs/1402.3319
Outline

• Trust networks
• Forming opinions with uncertainty
  – Subjective Logic
  – ... hard to combine with Trust Networks
• Subjective Logic revisited
  – new chaining rule
  – evidence plays central role
• Trust networks with the new chaining rule
Trust Networks

Question: what should user 1 think about proposition P?
Subjective Logic

Opinion triplet $x = (x_b, x_d, x_u)$

$x_b + x_d + x_u = 1$

Opinion about a proposition $P$, given some evidence $E$:

$x_b = \text{Prob}[ P \text{ can be proven } | E ]$
$x_d = \text{Prob}[ P \text{ can be disproven } | E ]$
$x_u = \text{Prob}[ \text{nothing can be proven about } P \ | \ E ]$
Subjective Logic $\neq$ fuzzy logic

Subjective Logic distinguishes between
- conflicting evidence
- lack of evidence
Relation between opinion and evidence

[Jøsang 2001]

\[ p = \text{amount of evidence supporting proposition } P \]
\[ n = \text{amount of evidence refuting } P \]

\[
(x_b, x_d, x_u) = \frac{(p, n, 2)}{p + n + 2}; \quad (p, n) = \frac{2(x_b, x_d)}{x_u}
\]

Strange constant; comes from Beta distributions ...
"Consensus" rule: Combining two opinions, \( x \) and \( y \)

\[
x \oplus y \overset{\text{def}}{=} \left( \frac{x_u y_b + y_u x_b, x_u y_d + y_u x_d, x_u y_u}{x_u + y_u - x_u y_u} \right)
\]

\[ p = p(x) + p(y) \quad n = n(x) + n(y) \]

Reduction of uncertainty
Chaining of opinions

\[ x \otimes y \overset{\text{def}}{=} (x_b y_b, x_b y_d, x_d + x_u + x_b y_u) \]

simple multiplication factor \(x_b\)

Associative: \(x \otimes (y \otimes z) = (x \otimes y) \otimes z\)
Problems with the chaining rule

• no interpretation in terms of evidence
  – pathological cases

• causes trouble in trust networks
  – double counting of evidence
  – topology-dependent formulas
  – cannot handle loops
    ("canonical form" needs loop removal)

• distributive property is missing,
  \[ x \otimes (y \oplus z) \neq (x \otimes y) \oplus (x \otimes z) \]

Underlying reason:
- Consensus is based on addition of evidence;
- Discounting based on probability multiplication.
Our contributions

• Relation between evidence and opinions
  – simpler theory

• Scalar multiplication rule
  – scales amount of evidence

• New chaining rule "⊠"
  – evidence plays central role
  – $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
  – can deal with arbitrary Trust Networks
Opinions and evidence, revisited

**Theorem 4.1**: Let \( p \geq 0 \) be the amount of evidence that supports ‘belief’; let \( n \geq 0 \) be the amount of evidence that supports ‘disbelief’. Let \( x = (b, d, u) \) be the opinion based on the evidence. If we demand the following properties,

1) \( \frac{b}{d} = \frac{p}{n} \)

2) \( b + d + u = 1 \)

3) \( p + n = 0 \Rightarrow u = 1 \)

4) \( p + n \to \infty \Rightarrow u \to 0 \)

then the unique solution for \( x \) is

\[
x = (b, d, u) = \frac{(p, n, c)}{p + n + c} ; \quad (p, n) = c \frac{(b, d)}{u}
\]

- \( c > 0 \), more general than "2"
- "caution" parameter

Consensus operation does not depend on \( c \)
Scalar multiple of an opinion

Definition of $\alpha \cdot x$:

\[
p(\alpha \cdot x) = \alpha \, p(x) \quad n(\alpha \cdot x) = \alpha \, n(x)
\]

\[
\alpha \cdot x \overset{\text{def}}{=} \frac{(\alpha x_b, \alpha x_d, x_u)}{\alpha(x_b + x_d) + x_u}
\]

Nice linear properties,

\[
\alpha \cdot (x \oplus y) = (\alpha \cdot x) \oplus (\alpha \cdot y) \quad \text{and} \quad (\alpha + \beta) \cdot x = (\alpha \cdot x) \oplus (\beta \cdot x)
\]

\[
\alpha \cdot (\beta \cdot x) = (\alpha \beta) \cdot x
\]
Choose a function $g$ that maps opinions to $[0,1)$

Definition: $x \boxtimes y = g(x) \cdot y$ [Scalar mult. of the evidence underlying $y$.]

Distribution property:

$$x \boxtimes (y \oplus z) = (x \boxtimes y) \oplus (x \boxtimes z)$$

But not associative. (Not necessary!)
Trust networks with the new chaining rule

- "Lossy transport of evidence"
- Recursive solution \( R_i \),
  given direct opinions \( A_{ij} \):

\[
R_{ij} = A_{ij} \oplus \bigoplus_{k : k \neq i} (R_{ik} \boxtimes A_{kj})
\]

Example: \( R_{16} = (R_{14} \boxtimes A_{46}) \oplus (R_{15} \boxtimes A_{56}) \)
\[ F_{1P} = R_{17} \boxtimes T_{7P} \]
\[ R_{17} = R_{16} \boxtimes A_{67} \]
\[ R_{16} = (R_{14} \boxtimes A_{46}) \oplus (R_{15} \boxtimes A_{56}) \]
\[ R_{15} = (R_{14} \boxtimes A_{45}) \oplus (R_{13} \boxtimes A_{35}) \]
\[ R_{14} = R_{13} \boxtimes A_{34} \]
\[ R_{13} = R_{12} \boxtimes A_{23} \]
\[ R_{12} = A_{12}. \]

\[ R_{16} = ((A_{12} \boxtimes A_{23}) \boxtimes A_{34}) \boxtimes A_{46} \oplus \left\{ ((A_{12} \boxtimes A_{23}) \boxtimes A_{34}) \boxtimes A_{45} \oplus (A_{12} \boxtimes A_{23}) \boxtimes A_{35} \right\} \boxtimes A_{56} \]
Numerical experiments

<table>
<thead>
<tr>
<th>Flow-based</th>
<th>Flow-SL</th>
<th>SL (canonical form)</th>
<th>EBSL ( g(x) = x_b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>0.401</td>
<td>(0.024, 0.220, 0.756)</td>
<td>(0.014, 0.123, 0.863)</td>
</tr>
<tr>
<td>C2</td>
<td>0.392</td>
<td>(0.003, 0.246, 0.751)</td>
<td>(0.002, 0.131, 0.861)</td>
</tr>
<tr>
<td>C3</td>
<td>0.501</td>
<td>(0.9993, 0.000, 0.0007)</td>
<td>(0.999, 0.000, 0.001)</td>
</tr>
</tbody>
</table>

Table 2: Comparison in terms of trust values \( r_{1P} \) and opinions \( F_{1P} \)

New chaining rule preserves more evidence
• Subjective Logic
  – good way of capturing uncertainty
  – addition rule $\oplus$ piles up evidence
  – "multiplication" rule $\otimes$ inconsistent with $\oplus$
    => all kinds of trouble

• Relation between evidence and opinions

• New: scalar multiple of opinion
  – scalar multiple of evidence

• New chaining rule $\triangleright$
  – no longer associative
  – (right) distribution property
  – arbitrary Trust Networks can be handled;
    lossy transport of evidence